HYPERSONIC FLOW PAST A CIRCULAR CONE AT ANGLE OF ATTACK

(GIPERZVUKOVOE OBTEKANIE KRUGLOGO KONUSA POD UGLOM ATAKI)

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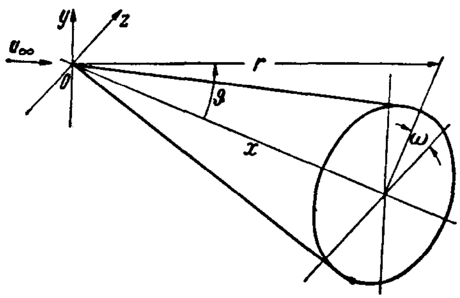
Cheng [1] has solved the problem of flow past a cone in a hypersonic stream of gas at angle of attack by the method of expansion in the small parameters $\varepsilon = (\gamma - 1)/(\gamma + 1)$ and $\sigma = \sin\alpha/\sin\tau$, where γ is the ratio of specific heats, α the angle of attack, and τ the semi-vertex angle of the cone. In the present note we solve the same problem, by the same method. We obtain the first approximation for the solution and its correction in the vicinity of the cone surface, and the second approximation for the pressure. However, the results are different from those of [1].



1. We consider flow past a circular cone of semi-vertex angle τ of a uniform hypersonic gas stream at angle of attack α, using a spherical coordinate system r, ϑ, ω with the axis coinciding with the axis of the cone (see figure).

We denote by u_+, v_+, w_+ the components of the velocity vector in the directions of increasing r, ϑ, ω and by p_+ and ρ_+ the pressure and density. We introduce dimensionless variables in the form

$$u \equiv \frac{u_+}{u_\infty}, \qquad v \equiv \frac{v_+}{\varepsilon u_\infty \sin\tau}, \qquad w \equiv \frac{w_+}{u_\infty \sin\alpha}$$
$$p \equiv \frac{p_+}{\rho_\infty u_\infty^2 \sin^2\tau}, \qquad \rho \equiv \frac{\varepsilon\rho_+}{\rho_\infty}, \qquad \theta \equiv \frac{\sin\vartheta - \sin\tau}{\varepsilon \sin\tau} \qquad (1.1)$$

We retain the notation of [1]. The equations of momentum, continuity, and energy take the form

$$\begin{gathered}
\left[Iv\frac{\partial}{\partial\theta}+\sigma\frac{w}{1+\varepsilon\theta}\frac{\partial}{\partial\omega}\right]u=\sin^2\tau\,[\varepsilon^2v^2+\sigma^2w^2]\\
\frac{\partial p}{\partial\theta}=\sigma^2\rho\frac{w^2}{1+\varepsilon\theta}-\varepsilon\rho\left[v\frac{\partial}{\partial\theta}+\sigma\frac{w}{I(1+\varepsilon\theta)}\frac{\partial}{\partial\omega}+\frac{u}{I}\right]v\\
\sigma\rho\left[IV\frac{\partial}{\partial\theta}+\sigma\frac{w}{1+\varepsilon\theta}\frac{\partial}{\partial\omega}+u\right]w=-\frac{\varepsilon}{1+\varepsilon\theta}\left[\frac{\partial p}{\partial\omega}+\sigma I\rho vw\right]\\
2(1+\varepsilon\theta)\rho u+I\frac{\partial}{\partial\theta}[(1+\varepsilon\theta)\rho v]+\sigma\frac{\partial}{\partial\omega}(\rho w)=0\\
\left[Iv\frac{\partial}{\partial\theta}+\sigma\frac{w}{1+\varepsilon\theta}\frac{\partial}{\partial\omega}\right]\left(\frac{p}{\rho^\gamma}\right)=0
\end{gathered}\tag{1.2}$$

Here

$$I=\cos\vartheta=[1-\sin^2\tau\,(1+\varepsilon\theta)^2]^{1/2}$$

At the surface of the cone we have the tangency condition

$$v=0\quad\text{for}\quad\theta=0$$

On the surface of the shock wave $\theta=\theta^+(\omega)$ we have the conditions of conservation of mass

$$I\,[\rho v-I\sigma\sin\omega+(1+\varepsilon\theta^+)\cos\alpha]=\sigma\frac{\theta_\omega^{\ +}}{1+\varepsilon\theta^+}[\rho w-\varepsilon\cos\omega]$$

of momentum

$$\begin{aligned}
\left[I^2+\left(\frac{\varepsilon\theta_\omega^{\ +}}{1+\varepsilon\theta^+}\right)^2\right](p-k\varepsilon)=\left[I(1+\varepsilon\theta^+)\cos\alpha-I^2\sigma\sin\omega+\sigma\varepsilon\frac{\theta_\omega^{\ +}}{1+\varepsilon\theta^+}\cos\omega\right]^2-\\
-\varepsilon\rho\left[Iv-\sigma\frac{\theta_w^{\ +}}{1+\varepsilon\theta^+}w\right]^2
\end{aligned}\tag{1.3}$$

of energy

$$\begin{aligned}
\left[I^2+\left(\frac{\varepsilon\theta_\omega^{\ +}}{1+\varepsilon\theta^+}\right)^2\right]\left(\frac{p}{\rho}-k\right)(1+\varepsilon)+\varepsilon^2\left[Iv-\sigma\frac{\theta_\omega^{\ +}}{1+\varepsilon\theta^+}w\right]^2=\\
=\left[I(1+\varepsilon\theta^+)\cos\alpha-I^2\sigma\sin\omega+\sigma\varepsilon\frac{\theta_\omega^{\ +}}{1+\varepsilon\theta^+}\cos\omega\right]^2
\end{aligned}$$

and of the tangential components

$$\begin{gathered}
I\sigma(w-\cos\omega)+\varepsilon\frac{\theta_\omega^{\ +}}{1+\varepsilon\theta^+}[\varepsilon V-I\sigma\sin\omega+(1+\varepsilon\theta^+)\cos\alpha]=0\\
u-I\cos\alpha=\sin^2\tau\,(1+\varepsilon\theta^+)\,\sigma\sin\omega
\end{gathered}$$

Here

$$k = \frac{\gamma + 1}{\gamma(\gamma - 1) M_\infty^2 \sin^2 \tau}, \qquad \theta_\omega^+ = \frac{d\theta^+(\omega)}{d\omega}$$

2. We seek a solution as a series in ε and σ of the form

$$p = p_{00} + p_{10}\varepsilon + p_{01}\sigma + p_{20}\varepsilon^2 + p_{11}\varepsilon\sigma + p_{02}\sigma^2 + \cdots \tag{2.1}$$

We also seek the function $\theta^+(\omega)$ that describes the location of the shock wave in the form

$$\theta^+ = \theta_{00} + \theta_{10}\varepsilon + \theta_{01}\sigma + \theta_{20}\varepsilon^2 + \theta_{11}\varepsilon\sigma + \theta_{02}\sigma^2 + \cdots \tag{2.2}$$

The quantity k is assumed to be bounded.

Substituting expansions such as (2.1) and (2.2) into the equations and boundary conditions, we collect terms with like powers of ε and σ, and obtain equations for the coefficients of the series (2.1) and (2.2). The solutions of the resulting equations are found in terms of elementary functions. We give the solution for the first approximation

$$\begin{aligned}
u &= \cos\tau - \varepsilon\left(\frac{1+k}{2}\right)\sin\tau\tan\tau + \sigma\sin^2\tau\sin\omega \\
v &= -2\theta + \varepsilon\left[\theta(1+k)\tan^2\tau + \theta^2(1 - \tan^2\tau) - \frac{4\theta^3}{3(1+k)}\right] + \\
&\quad + \sigma\frac{\sin\omega}{\cos\tau}\left[\frac{1+k}{3}\left(\frac{2\theta}{1+k}\right)^{3/2} - 2\theta\sin^2\tau\right] \\
w &= \cos\omega\left(\frac{2\theta}{1+k}\right)^{1/2} + \varepsilon\cos\omega\left\{2(1+k) + \left(\frac{2\theta}{1+k}\right)^{1/2}\left[\frac{k^2}{2(1+k)} - k + \right.\right. \\
&\quad \left. + \frac{1+k}{3\cos^2\tau} - \frac{15}{8}(1+k) - \frac{1+k}{4}\tan^2\tau\right] + \left(\frac{2\theta}{1+k}\right)^{3/2}\left[\frac{1+k}{8}\tan^2\tau - \frac{3}{8}(1+k)\right] - \\
&\quad \left. - \left(\frac{2\theta}{1+k}\right)^{5/2}\frac{1+k}{24}\right\} + \sigma\frac{\sin 2\omega}{\cos\tau}\left[\frac{1}{2}\left(1 - \frac{k}{1+k}\cos^2\tau\right)\left(\frac{2\theta}{1+k}\right)^{1/2} - \frac{1}{3}\left(\frac{2\theta}{1+k}\right)\right] \\
p &= 1 + \varepsilon\left(\frac{1+5k}{4} - \frac{\theta^2}{1+k}\right) - \sigma 2\cos\tau\sin\omega \\
\rho &= \frac{1}{1+k} + \varepsilon\left[\frac{1}{4} + \left(\frac{k}{1+k}\right)^2 - \left(\frac{\theta}{1+k}\right)^2\right] - \sigma\frac{2k}{(1+k)^2}\cos\tau\sin\omega \\
\theta^+ &= \frac{1+k}{2} + \varepsilon\left[\frac{(1+k)^2}{24}(7 + 3\tan^2\tau) - \frac{k^2}{2}\right] + \sigma\frac{\sin\omega}{\cos\tau}\left[k\cos^2\tau - \frac{1+k}{3}\right]
\end{aligned} \tag{2.3}$$

Higher-order approximations are found without difficulty in the same way. In particular, the second approximation for the pressure has the form

$$\begin{aligned}
p &= 1 + \varepsilon\left(\frac{1+5k}{4} - \frac{\theta^2}{1+k}\right) - \sigma 2\cos\tau\sin\omega + \varepsilon^2\left[\frac{3}{32}(1+k)^2 - \frac{k^2}{4} + \right. \\
&\quad + \frac{\tan^2\tau}{4}(1+k)^2 + \frac{(2\theta)^2}{4}\tan^2\tau - \left(\frac{\theta k}{1+k}\right)^2 - \frac{(2\theta)^2}{16} + \frac{5(2\theta)^3}{24(1+k)} -
\end{aligned}$$

$$- \frac{(2\theta)^3 \tan^2 \tau}{8(1+k)} - \frac{11}{96} \frac{(2\theta)^4}{(1+k)^2} \Big] + \sigma^2 \Big[\cos 2\tau - \cos^2 \omega \times \tag{2.4}$$

$$\times \left(\cos^2 \tau + \frac{1}{4} - \frac{\theta^2}{(1+k)^2} \right) \Big] + \sigma\varepsilon \frac{\sin \omega}{\cos \tau} \left\{ \frac{4}{15} (1+k) \left[\left(\frac{2\theta}{1+k} \right)^{5/2} - 1 \right] + \right.$$

$$\left. + \frac{\sin^2 \tau}{2} \left[1 - \frac{(20)^2}{1+k} \right] + \frac{k}{2} \left[\left(\frac{20}{1+k} \right)^2 \cos^2 \tau + 1 \right] \right\}$$

From (2.4) we obtain the coefficients of normal and longitudinal force

$$C_N = \frac{2N}{\rho_\infty u_\infty^2 \pi r^2 \sin^2 \tau} = \sin \alpha \left\{ 2 \cos^2 \tau + \varepsilon \left[\frac{4}{15} (1+k) - \frac{\sin^2 \tau}{2} - \frac{k}{2} \right] \right\}$$

$$C_X = \frac{2X}{\rho_\infty u_\infty^2 \pi r^2 \sin^2 \tau} = 2 \sin^2 \tau \left\{ 1 + \varepsilon \left(\frac{1+5k}{4} \right) + \right. \tag{2.5}$$

$$\left. + \sigma^2 \left(\frac{3\cos^2 \tau}{2} - \frac{9}{8} \right) + \varepsilon^2 \left[\frac{3}{32} (1+k)^2 - \frac{k^2}{4} + \frac{\tan^2 \tau}{4} (1+k)^2 \right] \right\}$$

3. The entropy calculated from the results of the first approximation assumes non-constant values at the surface of the cone, which means that the solution obtained by the given method is not valid in the vicinity of the cone surface [2,3].

It is shown in [1] that to a first approximation the entropy can be represented by

$$S = S_0 + \varepsilon S_{10} + \sigma S_{01}(\zeta) + O(\sigma^2)$$

Here

$$\zeta = \theta^{\sigma\varepsilon (1+k) \sec \tau} \tan \left(\frac{1}{2} \omega + \frac{1}{4} \pi \right) \tag{3.1}$$

The constants S_0, S_{10} and the function $S_{01}(\zeta)$ can be found from the boundary conditions for the entropy. Calculation gives

$$\frac{p}{\rho^\gamma} = 1 + k + \varepsilon [k + 2(1+k) \ln (1+k)] + 2\sigma \cos \tau \frac{1-\zeta^2}{1+\zeta^2} \tag{3.2}$$

As shown in [3], the expressions for w_0 and v_0 are valid even in the vicinity of the cone surface. The expression (2.4) for the pressure p is valid to the first order of small quantities. Then from (3.2) and (2.3) we have

$$\rho = \frac{1}{1+k} + \varepsilon \left[\frac{1}{4} + \left(\frac{k}{1+k} \right)^2 - \left(\frac{\theta}{1+k} \right)^2 \right] - \frac{2\sigma \cos \tau}{(1+k)^2} \left[\frac{1-\zeta^2}{1+\zeta^2} + (1+k) \sin \omega \right] \tag{3.3}$$

Using the Bernoulli integral we can obtain

$$u = \cos\tau - \frac{\varepsilon}{2}(1+k)\sin\tau\tan\tau - \sigma\sin^2\tau\,\frac{1-\zeta^2}{1+\zeta^2} \tag{3.4}$$

Outside the vicinity of the cone surface the expressions (3.3) and (3.4) go over to their counterparts in (2.3). Thus we obtain the first approximation to the solution of the problem of flow past a circular cone at angle of attack in a hypersonic gas stream. In the expressions obtained here for the density, shock wave shape, peripheral velocity, and pressure, terms in ln $(1 + k)$ are absent; they thus differ basically from the results of [1], and at the same time this term is present in the expression for entropy whereas it is absent from the result of Cheng.

In the special case $\sigma = 0$ the first approximation obtained corresponds to the result of Chernyi [4]. The error in Cheng's result probably arose from incorrect calculation of the entropy.

In conclusion I thank B.M. Bulakh for posing the problem and discussing the results.

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